Stability properties for the higher dimensional catenoid in $\mathbb{R}^{n+1}$

Luen-Fai Tam* and Detang Zhou†

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Abstract

This paper concerns some stability properties of higher dimensional catenoids in $\mathbb{R}^{n+1}$ with $n \geq 3$. We prove that higher dimensional catenoids have index one. We use δ-stablity for minimal hypersurfaces and show that the catenoid is $\frac{2}{n}$-stable and a complete $\frac{2}{n}$-stable minimal hypersurface is a catenoid or a hyperplane provided the second fundamental form satisfies some decay conditions.

Keywords: catenoid, minimal hypersurface, stability.
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1 Introduction

The catenoid in $\mathbb{R}^3$ is the only minimal surface of revolution other than the plane. So it can be regarded as the simplest minimal surface other than the plane. This motivates us to study higher dimensional catenoids as complete minimal hypersurfaces in higher dimensional Euclidean spaces $\mathbb{R}^{n+1}$, $n \geq 3$. In particular, we want to discuss some stability properties of the catenoids. Let us recall and introduce some notions of stability.

Let M^n be a minimal hypersurface in $\mathbb{R}^{n+1}$. M is said to be *stable* if

$$\int_M \left(|\nabla f|^2 - |A|^2 f^2\right) \geq 0 \tag{1.1}$$

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for all $f \in C_0^\infty(M)$, where $|A|$ is the norm of the second fundamental form of M. M is said to be *weakly stable* if (1.1) is true for all $f \in C_0^\infty(M)$ with $\int_M f = 0$, see [CCZ]. Recall that in [CM] it is defined that M is δ-*stable* if

$$\int_M \left(|\nabla f|^2 - (1-\delta)|A|^2 f^2\right) \geq 0 \tag{1.2}$$

for all $f \in C_0^\infty(M)$.

It is easy to see that M is stable implies that M is weakly stable and δ stable when $\delta \geq 0$. If M is a minimal surface in $\mathbb{R}^3$, then M is always δ-stable. By [FS], M is stable if and only if it there is a positive solution u of $(\Delta + |A|^2)u = 0$. Hence M is stable implies that the universal cover of M is stable. Similarly, M is δ-stable implies that the universal cover of M is also δ-stable. However, this is not true for weakly stable minimal hypersurface.

In $\mathbb{R}^3$, the catenoid is not stable by [Ln]. In $\mathbb{R}^{n+1}(n \geq 3)$, it is proved in [CSZ] that a complete stable minimal hypersurface must have only one end. So a catenoid in $\mathbb{R}^{n+1}(n \geq 3)$ is not stable since it has two ends. In fact, it is not even weakly stable (see [CCZ]). It is an interesting question to find the index of catenoids, which measures the degree of instability. Using the Gauss map, it was proved in [Fc, p.131-132] that catenoids in $\mathbb{R}^3$ have index 1. It is known that a complete minimal surface in $\mathbb{R}^3$ has finite index if and only if it has finite total curvature, see [Fc]. In [Sc], Schoen proved that the only complete nonflat embedded minimal surfaces in $\mathbb{R}^3$ with finite total curvature and with two ends are the catenoids. It was also proved in [LR] that only index one complete minimal surfaces are the catenoid and Enneper surface and catenoid is the only embedded minimal surface with index 1. Although it has been believed that a higher dimensional catenoid also has index one, we have not found a reference for a proof. The idea of using Gauss map in [Fc] does not work for higher dimension. In this work, using a different method we prove that the index of higher dimensional cateniod is indeed one, see Theorem 2.1. We would like to point out Choe [Ch] has constructed higher dimensional Enneper's hypersurfaces in $\mathbb{R}^{n+1}$ when $n = 3, 4, 5, 6$. Different from the catenoid, we don't even know whether it is of finite index.

It is well known that for $2 \leq n \leq 6$, a complete area minimizing hypersurface in $\mathbb{R}^{n+1}$ must be a hyperplane. It is well known by a result of do Carmo-Peng [dCP1] and Fischer-Colbrie-Schoen [FS] independently that a complete stable minimal surface in $\mathbb{R}^3$ is a plane. On the other hand, for $3 \leq n \leq 6$ it is still an open question whether the condition of area minimizing can be replaced by stability. In this direction, it was proved in [dCP2] (see

also [dCD]) that a complete stable minimal hypersurface in $\mathbb{R}^{n+1}$ is indeed a hyperplane under some additional assumptions, for example: the norm of the second fundamental form is square integrable. We will prove a similar result for catenoids which states that a complete $\frac{2}{n}$-stable minimal hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 3$ is a catenoid if the norm of the second fundamental form satisfies certain decay conditions. See Theorem 4.1. As a corollary to this, we show that if a $\frac{2}{n}$-stable complete proper immersed minimal hypersurface M^n in $\mathbb{R}^{n+1}$ with $n \geq 3$ has least area outside a compact set, and if the norm of the second fundamental form is square integrable then M is either a hyperplane or a catenoid.

The paper is organized as follows: in §2, we introduce the definition and discuss some general properties of catenoids in higher dimensional Euclidean spaces. We will also prove that catenoids have index one. In §3, we use the Simons' computation and the result in [dCD] to give a characterization of catenoids. In §4, we will discuss $\frac{2}{n}$-stability and catenoids.

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2 Catenoid and its index

In this section, we will recall the definition of catenoid, show that it is $\frac{2}{n}$-stable and compute its index. Following do Carmo and Dajczer [dCD], a catenoid is a complete rotation minimal hypersurface in $\mathbb{R}^{n+1}$, $n \geq 2$ which is not a hyperplane. More precisely, let $\phi(s)$ be the solution of

$$\begin{cases} \frac{\phi''}{(1+\phi'^2)^{\frac{3}{2}}} - \frac{n-1}{\phi(1+\phi'^2)^{\frac{1}{2}}} = 0; \\ \phi(0) = \phi_0 > 0; \\ \phi'(0) = 0. \end{cases} \tag{2.1}$$

ϕ can be obtained as follows. Consider

$$s = \int_{\phi_0}^{\phi} \frac{d\tau}{(a\tau^{2(n-1)} - 1)^{\frac{1}{2}}} \tag{2.2}$$

where $a = \phi_0^{-2(n-1)}$. The integral in the right side of (2.2) is defined for all $\phi \geq \phi_0$. The function $s(\phi)$ is increasing and if $n = 2$, it maps $[\phi_0, \infty)$ onto

$[0,\infty)$; if $n \geq 3$, then it maps $[\phi_0, \infty)$ onto $[0, S)$ where

$$S = S(\phi_0) = \int_{\phi_0}^{+\infty} \frac{d\tau}{(a\tau^{2(n-1)} - 1)^{\frac{1}{2}}} < \infty.$$

So $\phi(s)$ can be defined, and it is smooth up to 0 such that $\phi'(0) = 0$. If we extend ϕ as an even function, then ϕ is smooth and satisfies (2.1) on $\mathbb{R}$ in case $n = 2$ and on $(-S, S)$ in case $n \geq 3$.

Let $\mathbb{S}^{n-1}$ be the standard unit sphere in $\mathbb{R}^n$. A point $\omega \in \mathbb{S}^{n-1}$ can also be considered as the unit vector ω in $\mathbb{R}^n$ which in turn is identified as the hyperplane $x_{n+1} = 0$ in $\mathbb{R}^{n+1}$.

Definition 2.1 *A catenoid in $\mathbb{R}^{n+1}$ is the hypersurface defined by the embedding:*

$$F : I \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$$

with $F(s, \omega) = (\phi(s)\omega, s)$, where $I = \mathbb{R}$ if $n = 2$ and $I = (-S(\phi_0), S(\phi_0))$ if $n \geq 3$, and ϕ is the solution of (2.1).

A hypersurface obtained by a rigid motion of the hypersurface in the definition will also called a catenoid. In case $n = 2$, this is the standard catenoid in $\mathbb{R}^3$. From now on, we are interested in the case that $n \geq 3$.

Proposition 2.1 *Let M be a catenoid in $\mathbb{R}^{n+1}$ as in Definition 2.1, $n \geq 3$. We have:*

(i) M is complete.

(ii) The principal curvatures are $\lambda_1 = -\frac{\phi''}{(1+\phi'^2)^{\frac{3}{2}}}$, $\lambda_2 = \cdots = \lambda_n = \frac{1}{\phi(1+\phi'^2)^{\frac{1}{2}}}$.

(iii) M is minimal.

(iv) The norm $|A|$ of the second fundamental form A of M is nowhere zero. Moreover, $|A|$ satisfies

$$|A|\Delta|A| + |A|^4 = \frac{2}{n}|\nabla|A||^2. \tag{2.3}$$

(v) M is symmetric with respect to the hyperplane $x_{n+1} = 0$ and is invariant under $O(n)$ which is the subgroup of orthogonal transformations on $\mathbb{R}^{n+1}$ which fix the x_{n+1} axis.

(vi) The part $\{x \in M|\ x_{n+1} \geq 0\}$ and the part $\{x \in M|\ x_{n+1} \leq 0\}$ are graphs over a subset of $\{x_{n+1} = 0\}$.

(vii) Let P be a hyperplane containing the x_{n+1} axis. Then P divides M into two parts, each is a graph over P.

Proof : (i), (v), (vi) and (vii) are immediate consequences of the definition. Let $N = \frac{1}{(1+\phi'^2)^{\frac{1}{2}}}(\omega, -\phi')$. Here and below $'$ and $''$ are derivatives with respect to s. Then N is the unit normal of M. Let D be the covariant derivative operator in $\mathbb{R}^{n+1}$. Then

$$D_{\frac{\partial}{\partial s}} N = -\frac{\phi''}{(1+\phi'^2)^{\frac{3}{2}}}(\phi'\omega, 1) = -\frac{\phi''}{(1+\phi'^2)^{\frac{3}{2}}}\frac{\partial}{\partial s}.$$

Suppose $(t_1, \dots, t_{n-1})$ are local coordinates of $\mathbb{S}^{n-1}$, then

$$D_{\frac{\partial}{\partial t_i}} N = \frac{1}{(1+\phi'^2)^{\frac{1}{2}}}(\frac{\partial}{\partial t_i}\omega, 0) = \frac{1}{\phi(1+\phi'^2)^{\frac{1}{2}}}(\phi\frac{\partial}{\partial t_i}\omega, 0) = \frac{1}{\phi(1+\phi'^2)^{\frac{1}{2}}}\frac{\partial}{\partial t_i}.$$

From these (ii) follows.

(iii) follows from (ii) and (2.1).

(iv) First note that (2.2) implies $\phi' = (a\phi^{2(n-1)} - 1)^{\frac{1}{2}}$, then,

$$\begin{aligned} |A|^2 &= \frac{n(n-1)}{\phi^2(1+\phi'^2)} \\ &= n(n-1)\phi_0^{2(n-1)}\phi^{-2n}. \end{aligned} \tag{2.4}$$

Hence $|A| > 0$ everywhere because $\phi \geq \phi_0 > 0$. On the other hand, the metric on M in the coordinates s, ω is given by

$$g = (1+\phi'^2)ds^2 + \phi^2 g_{\mathbb{S}^{n-1}} \tag{2.5}$$

where $g_{\mathbb{S}^{n-1}}$ is the standard metric on $\mathbb{S}^{n-1}$. Then

$$\begin{aligned} \Delta\phi &= (1+\phi'^2)^{-1}\phi'' + \left[(1+\phi'^2)^{-1}\right]'\phi' + (1+\phi'^2)^{-1}\phi'\left[\log\left((1+\phi'^2)^{\frac{1}{2}}\phi^{n-1}\right)\right]' \\ &= (1+\phi'^2)^{-1}\phi'' - 2(1+\phi'^2)^{-2}\phi'^2\phi'' + (1+\phi'^2)^{-1}\phi'\left[\frac{\phi'\phi''}{1+\phi'^2} + \frac{(n-1)\phi'}{\phi}\right] \\ &= \frac{\phi''}{(1+\phi'^2)^2} + (n-1)\frac{|\nabla\phi|^2}{\phi} \\ &= (n-1)\frac{1}{\phi(1+\phi'^2)} + (n-1)\frac{|\nabla\phi|^2}{\phi} \end{aligned} \tag{2.6}$$

where ∇ is the covariant derivative of M and we have used (2.1). (2.3) follows from (2.4) and (2.6) by a direct computation. $\square$

By (iv) of the Proposition, we see that

$$\Delta|A|^{\frac{n}{n-2}} + \frac{n-2}{n}|A|^2|A|^{\frac{n}{n-2}} = 0$$

and $|A|^{\frac{n}{n-2}} > 0$. Hence the catenoid is $\frac{2}{n}$-stable by [Fc].

Theorem 2.1 *Let M be a catenoid in $\mathbb{R}^{n+1}$. Then index of M is 1.*

Proof : It is well known that M is not stable. One can also use the result of Cao, Shen and Zhu[CSZ]. They proved that any complete stable minimal hypersurfaces in $\mathbb{R}^{n+1}$ has only one end, since the catenoid has two ends, thus the index of M is at least 1. We only need to prove that its index is at most 1. Recall that the stability operator is written as

$$L = \Delta + |A|^2.$$

For M above $|A|^2(x)$ is an even function depending only on r. From the fact that M is unstable it follows that $\lambda_1(L) < 0$. we now show that the second eigenvalue $\lambda_2^D(L) \geq 0$ of L on any bounded domain $D \subset M$. Assume for the sake of contradiction that it were not true, we can find a domain $D(R) = (-R, R) \times \mathbb{S}^{n-1}$ such that $\lambda_2^{D(R)}(L) < 0$. Here $0 < R < S$ and $S = S(\phi_0)$ is as in Definition 2.1. That is to say that there is a function f satisfying

$$\begin{cases} Lf = -\lambda_2 f, & \text{in } D(R); \\ f|_{\partial D(R)} = 0. \end{cases} \tag{2.7}$$

We claim that f depends only on r. For any unit vector $v \in \mathbb{S}^n$, and $v \perp (1, 0, \cdots, 0)$, denote by π_v the hyperplane

$$\{p \in \mathbb{R}^{n+1}, \langle p, v\rangle = 0\}.$$

Let σ_v be the reflection with respect to π_v. Define function $\varphi_v(r,\theta) = f(r,\theta) - f_v(r,\theta)$ where $f_v(p) := f(\sigma_v(p))$ for any $p \in D(R)$. Since

$$\Delta f(r,\theta) = \frac{\partial^2 f}{\partial r^2} + \frac{a'(r)}{a(r)}\frac{\partial f}{\partial r} + \frac{1}{a^2(r)}\Delta_{S^{n-1}} f, \tag{2.8}$$

f_v also satisfies (2.8). Then

$$\begin{cases} L\varphi_v = -\lambda_2\varphi_v, & \text{in } D(R); \\ \varphi_v|_{D(R)\cap\pi_v} = 0. \end{cases} \tag{2.9}$$

Denote

$$D_v^+(R) := \{p \in \mathbb{R}^{n+1}, \langle p, v\rangle > 0\}.$$

Then $D_v^+(R)$ is a minimal graph over a domain in π_v thus is stable. From (2.9) and $\lambda_2 < 0$, we conclude that $\varphi_v \equiv 0$. It is a well-known fact that any element in orthogonal group $O(n-1)$ can be expressed as a composition of finite number of reflections, we know that f is rotationally symmetric.

Since f is the second eigenfunction of L, it changes sign, so there exists a number $r_0 \in (-R, R)$ such that $f(r_0) = 0$. Assume without loss of generality that $r_0 \geq 0$. We take $D(r_0, R) := \{p = (r, \theta) \in D(R), r \in (r_0, R)\}$. Again f is an eigenfunction of L on $D(r_0, R)$. Again we know that $D(r_0, R)$ is a minimal graph which contradicts the fact that $\lambda_2 < 0$ because f cannot be identically zero in $D(r_0, R)$. The contradiction shows the index of M is 1.$\square$

3 Simons' equation and catenoid

By Proposition 2.1, the norm of the second fundamental of a catenoid is nowhere zero and satisfies (2.3). In this section, we will prove that a complete non flat minimal hypersurface in $\mathbb{R}^{n+1}$ satisfying (2.3) must be a catenoid. Let us recall the Simons' computation on the second fundamental form of a minimal hypersurface in Euclidean space.

Let M be an n-dimensional manifold immersed in $\mathbb{R}^{n+1}$. Let A be its second fundamental form and ∇A be its covariant derivative. Let h_{ij} and h_{ijk} be the components of A and ∇A in an orthonormal frame.

By Proposition 2.1(iv), we see that Simons inequality becomes equality for catenoids. We will prove that the converse is also true. We first prove a lemma.

Lemma 3.1 *Let M be an immersed oriented minimal hypersurface in $\mathbb{R}^{n+1}$. At a point where the norm of the second fundamental form $|A| > 0$, we have*

$$|A|\Delta|A| + |A|^4 = \frac{2}{n}|\nabla|A||^2 + E. \tag{3.1}$$

with $E \geq 0$. Moreover, in an orthonormal frame e_i such that $h_{ij} = \lambda_i \delta_{ij}$, then $E = E_1 + E_2 + E_3$, where

$$\begin{cases} E_1 &= \sum_{j\neq i, k\neq i, k\neq j} h_{ijk}^2, \\ E_2 &= \frac{2}{n} \sum_{j\neq i, k\neq i, k\neq j} (h_{kki} - h_{jji})^2, \\ E_3 &= (1+\frac{2}{n})|A|^{-2} \sum_k \sum_{i\neq j} (h_{ii}h_{jjk} - h_{jj}h_{iik})^2. \end{cases} \tag{3.2}$$

Proof : At a point p where $|A| > 0$, choose an orthonormal frame such that $h_{ij} = \lambda_i \delta_{ij}$. Since M is minimal, then by [SSY, (1.20),(1.27)], for $|A| > 0$ we have:

$$|A|\Delta|A| + |A|^4 = \sum_{i,j,k=1}^{n} h_{ijk}^2 - |\nabla |A||^2. \tag{3.3}$$

Now,

$$\begin{aligned} |\nabla |A||^2 &= \left[\sum_k (\sum_i h_{ii} h_{iik})^2 \right] |A|^{-2} \\ &= \left[\sum_k (\sum_i h_{ii}^2 \sum_i h_{iik}^2) - \sum_k \sum_{i\neq j} (h_{ii}h_{jjk} - h_{jj}h_{iik})^2 \right] |A|^{-2} \\ &= \sum_{k,i} h_{iik}^2 - \left[\sum_{i\neq j} (h_{ii}h_{jjk} - h_{jj}h_{iik})^2 \right] |A|^{-2} \end{aligned} \tag{3.4}$$

where we have used the fact that M is minimal. On the other hand,

$$\begin{aligned} \sum_{k,i} h_{iik}^2 &= \sum_{k\neq i} h_{iik}^2 + \sum_i h_{iii}^2 \\ &= \sum_{k\neq i} h_{iik}^2 + \sum_i (\sum_{j\neq i} h_{jji})^2 \\ &= \sum_{k\neq i} h_{iik}^2 + \sum_i \left[(n-1) \sum_{j\neq i} h_{jji}^2 - \sum_{j\neq i, k\neq i, k\neq j} (h_{kki} - h_{jji})^2 \right] \\ &= n \sum_{k\neq i} h_{iik}^2 - \sum_{j\neq i, k\neq i, k\neq j} (h_{kki} - h_{jji})^2 \end{aligned} \tag{3.5}$$

Combining this with (3.4)

$$\sum_{k\neq i} h_{iik}^2 = \frac{1}{n} \left[|\nabla |A||^2 + \left(\sum_{i\neq j} (h_{ii}h_{jjk} - h_{jj}h_{iik})^2 \right) |A|^{-2} + \sum_{j\neq i, k\neq i, k\neq j} (h_{kki} - h_{jji})^2 \right] \tag{3.6}$$

Note that since $\mathbb{R}^{n+1}$ is flat, we have $h_{ijk} = h_{ikj}$, (see [SSY, (1.13)] for example). By (3.4),

$$\begin{aligned}
&\sum_{i,j,k} h_{ijk}^2 - |\nabla|A||^2 \\
&\qquad = \sum_{j\neq i, k\neq i, k\neq j} h_{ijk}^2 + \sum_{i\neq k} h_{iik}^2 + \sum_{i\neq k} h_{iki}^2 + \sum_{i\neq k} h_{ikk}^2 + \sum_i h_{iii}^2 - |\nabla|A||^2 \\
&\qquad = \sum_{j\neq i, k\neq i, k\neq j} h_{ijk}^2 + 2\sum_{i\neq k} h_{iik}^2 + \sum_{i,k} h_{iik}^2 - |\nabla|A||^2 \\
&\qquad = \sum_{j\neq i, k\neq i, k\neq j} h_{ijk}^2 + 2\sum_{i\neq k} h_{iik}^2 + \left(\sum_{i\neq j}(h_{ii}h_{jjk} - h_{jj}h_{iik})^2\right)|A|^{-2}
\end{aligned} \tag{3.7}$$

(3.1) follows from (3.3), (3.6) and (3.7). $\square$

Since E are nonnegative, we have the following Simons inequality, see [SSY]:

$$|A|\Delta|A| + |A|^4 \geq \frac{2}{n}|\nabla|A||^2 \tag{3.8}$$

at the point $|A| > 0$.

Now we are ready to prove the following:

Theorem 3.1 *Let $M^n (n \geq 3)$ be a non-flat complete immersed minimal hypersurface in $\mathbb{R}^{n+1}$. If the Simons inequality (3.8) holds as an equation on all nonvanishing point of $|A|$ in M, then M must be a catenoid.*

Proof : Suppose $\Phi : M \to \mathbb{R}^{n+1}$ is the minimal immersion. Since M is not a hyperplane, then $|A|$ is a nonnegative continuous function which does not vanish identically. Let p be a point such that $|A|(p) > 0$. Then $|A| > 0$ in a connected open set U containing p. Suppose that $|\nabla|A|| \equiv 0$ in U, then $|A|$ is a positive constant in U. Since $|A|$ satisfies:

$$|A|\Delta|A| + |A|^4 = \frac{2}{n}|\nabla|A||^2$$

we have a contradiction. Hence there is a point in U such that $|\nabla|A|| \neq 0$. By shrinking U, we may assume that $|A| > 0$ and $|\nabla|A|| > 0$ in U. By (3.1) and the fact that (3.8) is an equality in U, we conclude that $E = 0$ in U.

Let $q \in U$. Choose an orthonormal frame at q so that the second fundamental form is diagonalized, $h_{ij} = \lambda_i \delta_{ij}$. $E_2 = 0$ implies

$$h_{jji} = h_{kki}, \text{ for all } j \neq i, \quad k \neq i.$$

Combining with the minimal condition, we have

$$h_{iii} = -(n-1)h_{jji}, \text{ for all } j \neq i. \tag{3.9}$$

Since $|\nabla|A|| \neq 0$, then there exist i_0 and $j_0 \neq i_0$ such that $h_{j_0 j_0 i_0} \neq 0$ hence $h_{i_0 i_0 i_0} \neq 0$. Suppose for simplicity that $i_0 = 1$.

$E_3 = 0$ implies

$$h_{ii}h_{jjk} = h_{jj}h_{iik}, \text{ for all } i, j, k,$$

then

$$h_{11}h_{jj1} = h_{jj}h_{111} = -(n-1)h_{jj}h_{jj1}, \text{ for all } j \neq 1,$$

by (3.9). So

$$h_{11} = -(n-1)h_{jj}, \text{ for all } j \neq 1 \tag{3.10}$$

because $-(n-1)h_{jj1} = h_{111} \neq 0$. Hence the eigenvalues of h_{ij} are λ with multiplicity $n-1$ and $-(n-1)\lambda$ with $\lambda \neq 0$ because $|A| > 0$. Hence in a neighborhood of of p the eigenvalues of h_{ij} are of this form. By a result of do Carmo and Dajczer [dCD, Corollary 4.4], this neighborhood is part of a catenoid. Hence $\Phi(M)$ is contained in a catenoid $\mathcal{C}$ by minimality of the immersion. Since M is complete and Φ is a local isometry into the catenoid $\mathcal{C}$ which is simply connected because $n \geq 3$, Φ must be an embedding, see [Sp, p.330]. Hence $\Phi(M)$ is the catenoid. $\square$

4 $\frac{2}{n}$-stability and catenoid

In this section, we will prove that a complete immersed minimal hypersurface in $\mathbb{R}^{n+1}$, $n \geq 3$ is a catenoid if it is $\frac{2}{n}$-stable and if the second fundamental form satisfies some decay conditions. We will also discuss the case when the minimal hypersurface is area minimizing outside a compact set.

Following [SSY], let M be a complete immersed minimal hypersurface in $\mathbb{R}^{n+1}$, $n \geq 3$. Assume there is a Lipschitz function $r(x)$ defined on M such that $|\nabla r| \leq 1$ a.e. Define $B(R)$ for $0 < R < \infty$ by

$$B(R) = \{x \in M |\ r(x) < R\}.$$

Assume also that $B(R)$ is compact for all R and $M = \bigcup_{R>0} B(R)$. For example, $B(R)$ may be an intrinsic geodesic ball or the intersection of an extrinsic ball with M. In the later case, we assume that M is proper.

Theorem 4.1 *Let $M^n (n \geq 3)$ be a $\frac{2}{n}$-stable complete immersed minimal hypersurface in $\mathbb{R}^{n+1}$. If*

$$\lim_{R\to+\infty} \frac{1}{R^2} \int_{B(2R)\setminus B(R)} |A|^{\frac{2(n-2)}{n}} = 0, \tag{4.1}$$

then M is either a plane or a catenoid.

Proof : For any $\epsilon > 0$, let $u := (|A|^2 + \epsilon)^{\frac{\alpha}{2}}$, where $\alpha = \frac{n-2}{n}$. Then at the point $|A| > 0$,

$$\begin{aligned}
\Delta u &= u\left(\Delta \log u + |\nabla \log u|^2\right) \\
&= \frac{\alpha u}{2}\left(\frac{\Delta|A|^2}{|A|^2+\epsilon} - \frac{|\nabla|A|^2|^2}{(|A|^2+\epsilon)^2}\right) + \frac{u\alpha^2}{4}\frac{|\nabla|A|^2|^2}{(|A|^2+\epsilon)^2} \\
&= \alpha u\left(\frac{\frac{1}{2}\Delta|A|^2}{|A|^2+\epsilon} + (\alpha-2)\frac{|A|^2|\nabla|A||^2}{(|A|^2+\epsilon)^2}\right) \\
&= \alpha u\left(\frac{(2-\alpha)|\nabla|A||^2 - |A|^4 + E}{|A|^2+\epsilon} + (\alpha-2)\frac{|A|^2|\nabla|A||^2}{(|A|^2+\epsilon)^2}\right) \\
&\geq -\alpha u|A|^2 + \frac{\alpha u E}{|A|^2+\epsilon}
\end{aligned} \tag{4.2}$$

where we have used (1.2) and $E = E_1 + E_2 + E_3 \geq 0$. If we extend E to be zero for $|A| = 0$, then it is easy to see that the above inequality is still true.

On the other hand, for any function $\phi \in C_o^\infty(M)$,

$$\begin{aligned}
\int_M \phi^2 \frac{\alpha u E}{|A|^2+\epsilon} &\leq \int_M \phi^2 u\left(\Delta u + \alpha|A|^2 u\right) \\
&= -\int_M \phi^2|\nabla u|^2 - 2\int_M \phi u\langle\nabla u, \nabla\phi\rangle + \int_M \alpha|A|^2\phi^2u^2 \\
&\leq -2\int_M \phi u\langle\nabla u, \nabla\phi\rangle - \int_M \phi^2|\nabla u|^2 + \int_M |\nabla(\phi u)|^2 \\
&= \int_M |\nabla\phi|^2 u^2.
\end{aligned} \tag{4.3}$$

Here we have used (1.2). Let ϕ be a smooth function on $[0,\infty)$ such that $\phi \geq 0$, $\phi = 1$ on $[0,R]$ and $\phi = 0$ in $[2R,\infty)$ with $|\phi'| \leq \frac{2}{R}$. Then consider $\phi \circ r$, where r is the function in the definition of $B(R)$.

$$\int_{B(R)} \phi^2 \frac{\alpha u E}{|A|^2+\epsilon} \leq \int_{B(R)} \phi^2 u \left(\Delta u + \alpha |A|^2 u\right) \leq \frac{4}{R^2} \int_{B(2R)\setminus B(R)} ||A|^{\frac{2(n-2)}{n}}. \tag{4.4}$$

Let $\epsilon \to 0$ and then let $R \to +\infty$, we conclude that $E = 0$ whenever $|A| > 0$. Thus the Simons' inequality becomes equality on $|A| > 0$. By Theorem 3.1, we know that it is a catenoid.

Remark 1 *It should be remarked that (4.1) is satisfied when M is a catenoid. In fact, using notation in the Definition 2.1, the metric is of the form*

$$g = (1+\phi'^2)ds^2 + \phi^2 g_{\mathbb{S}^{n-1}}.$$

Hence the distance function is of order ϕ. By (2.4), $|A|$ is of order ϕ^{-n}. The volume of geodesic ball of radius $r \sim \phi$ is of order ϕ^n. From this it is easy to see that (4.1) is true for $n \geq 3$.

We say that M has least area outside a compact set (see [SSY], p. 283]), if (i) M is proper; and (ii) M is the boundary of some open set U in $\mathbb{R}^{n+1}$ and there is $R_0 > 0$ such that for any open set $\mathcal{O}$ in $\mathbb{R}^{n+1}$ with $\mathcal{O} \cap \widetilde{B}(R_0) = \emptyset$ we have $|\partial U \cap \mathcal{O}| \leq |\partial \mathcal{O} \cap U|$. Here $\widetilde{B}(R_0)$ is the extrinsic ball in $\mathbb{R}^{n+1}$ with center at the origin. If this is true, then M is stable outside a compact set and if r is the extrinsic distance, then

$$|B(4R) \setminus B(\frac{1}{2}R)| \leq |\partial \widetilde{B}(4R)| + |\partial \widetilde{B}(\frac{1}{2}R)| \leq CR^n$$

if R is large.

Corollary 4.1 *Let M^n, $n \geq 3$ be a $\frac{2}{n}$-stable complete proper immersed minimal hypersurface in $\mathbb{R}^{n+1}$. If M has least area outside a compact set and*

$$\int_M |A|^p < \infty, \tag{4.5}$$

for some $\frac{2(n-2)}{n} \leq p \leq 2$ then M is either a plane or a catenoid.

Proof : Suppose $|A|$ satisfies (4.5). Since $\frac{2(n-2)}{n} \leq p \leq 2$, we have:

$$\begin{aligned} R^{-2}\int_{B(2R)\setminus B(R)} |A|^{\frac{2(n-2)}{n}} &\leq CR^{-2}\left(\int_M |A|^p\right)^{\frac{2(n-2)}{pn}} R^{n-\frac{2(n-2)}{p}} \\ &= C\left(\int_M |A|^p\right)^{\frac{2(n-2)}{pn}} R^{(n-2)(1-\frac{2}{p})} \to 0 \end{aligned} \tag{4.6}$$

as $R \to \infty$. The result follows from Theorem 4.1. $\square$

By [Sc], the only nonflat complete minimal immersions of $M^n \subset \mathbb{R}^{n+1}$, which are regular at infinity and have two ends, are the catenoids. By the corollary, we have the following:

Corollary 4.2 *A nonflat complete minimal immersion of $M^n \subset \mathbb{R}^{n+1}$ with $n \geq 3$, which are regular at infinity and has more than two ends, is not $\frac{2}{n}$-stable.*

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Luen-fai Tam
Department of Mathematics
The Chinese University of Hong Kong
Shatin, Hong Kong, China
e-mail: lftam@math.cuhk.edu.hk

Detang Zhou
Insitituto de Matematica
Universidade Federal Fluminense- UFF
Centro, Niterói, RJ 24020-140, Brazil
email: zhou@impa.br